# Classical solutions to quasilinear parabolic problems with dynamic boundary conditions *

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**Abstract**

We study linear nonautonomous parabolic systems with dynamic boundary conditions. Next, we apply these results to show a theorem of local existence and uniqueness of a classical solution to a second order quasilinear system with nonlinear dynamic boundary conditions.

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## 1 Introduction and preliminaries

The main aim of this paper is to study existence and uniqueness of local classical solutions to quasilinear parabolic systems with dynamic boundary conditions in the form

$$\begin{cases} D_t u(t,x) = \sum_{|\alpha|=2} a_\alpha(t,x,u(t,x),\nabla_x u(t,x)) D_x^\alpha u(t,x) + f(t,x,u(t,x),\nabla_x u(t,x)), & t \geq 0, x \in \Omega, \\ D_t u(t,x') + \sum_{j=1}^n b_j(t,x',u(t,x')) D_{x_j} u(t,x') = h(t,x,u(t,x)), & t \geq 0, x' \in \partial\Omega, \\ u(0,x) = u_0(x), & x \in \Omega. \end{cases} \tag{1.1}$$

With the term "classical solutions", we mean solutions possessing all the derivatives appearing in (1.1) in pointwise sense: so we look for conditions guaranteeing the existence of solutions $u(t,x)$ with $D_t u$, $D_x^\alpha u$ ($|\alpha| \leq 2$) continuous in $[0,\tau] \times \overline{\Omega}$, for some $\tau > 0$. If the functions $b_j$ and $h$ are suitably regular, these conditions, together with the second equation in (1.1) (the boundary condition) imply that the map $t \to D_t u(t,\cdot)$ should be continuous with values in $C^1(\partial\Omega)$. However, it is well known that, in order to get neat results for parabolic problems, it is often advisable to replace continuous with Hölder continuous functions. So a natural class of solutions could be the set of functions in $C^{1+\beta/2,2+\beta}((0,\tau) \times \Omega)$ (see (1.7)), such that the restriction of $D_t u$ to $[0,\tau] \times \partial\Omega$ is bounded (as a function of $t$) with values in $C^{1+\beta}(\partial\Omega)$.

In order to frame our results, I begin by recalling some previous literature, concerning nonlinear parabolic problems with dynamic boundary conditions. The first paper I quote is [9]: here the author considers the system

$$\begin{cases} D_t u - \sum_{j=1}^n D_{x_j}[a_j(u,\cdot,\nabla_x u) D_{x_j} u] + a_0(u,\cdot) = f(t,\cdot) & \text{in} \quad \mathbb{R}^+ \times \Omega, \\ D_t u + \sum_{j,k=1}^n a_{jk}(u,\cdot)\nu_j D_{x_k} u + b_0(u,\cdot) u = g_1(u) & \text{on} \quad \mathbb{R}^+ \times \Gamma_1, \\ \sum_{j,k=1}^n a_{jk}(u,\cdot)\nu_j D_{x_k} u + b_1(u,\cdot) = 0 & \text{on} \quad \mathbb{R}^+ \times \Gamma_2, \\ u(0,\cdot) = u_0 & \text{in} \quad \Omega, \end{cases} \tag{1.2}$$

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where I have indicated with $\nu(x')$ the unit normal vector to $\partial\Omega$ in $x' \in \partial\Omega$, pointing outside $\Omega$. Employing monotonicity assumptions and Rothe's method, imposing a polynomial growth of the coefficients, he constructs a generalized solution. The initial datum $u_0$ is taken in a suitable space $W^{1,p}(\Omega)$, with $p$ connected with the growth conditions of the functions $a_j$. Some results of regularity are proved (for example, the solution in continuos with values in $H^2(\Omega')$ for every $\Omega'$ with compact closure in $\Omega$).

In the paper [3], the author considers the system

$$\begin{cases} D_t u - \sum_{j,k=1}^n D_{x_j}[a_{jk}(u,\cdot)D_{x_k}u] + \sum_{j=1}^n a_j(u,\cdot)D_{x_j}u + a_0(u,\cdot)u = f(u) & \text{in} \quad \mathbb{R}^+ \times \Omega, \\ \epsilon D_t u + \sum_{j,k=1}^n a_{jk}(u,\cdot)\nu_j D_{x_k}u + b_0(u,\cdot) = g_1(u) & \text{on} \quad \mathbb{R}^+ \times \Gamma_1, \\ \sum_{j,k=1}^n a_{jk}(u,\cdot)\nu_j D_{x_k}u + b_0(u,\cdot)u = g_1(u) & \text{on} \quad \mathbb{R}^+ \times \Gamma_2, \\ u(0,\cdot) = u_0 & \text{in} \quad \Omega. \end{cases} \tag{1.3}$$

Here $u : [0,\infty) \times \Omega \to \mathbb{R}^N$ ($a_{jk}(u,\cdot)$, $a_0(u,\cdot)$, $b_0(u,\cdot)$ are, in fact, matrixes), $\sum_{j,k=1}^n D_{x_j}[a_{jk}(\xi,\cdot)D_{x_k}]$ are normally elliptic and all the functions appearing in the first equation in (1.3) are smooth ($C^\infty$). If $p > n$, $n/p < r < 1 < \tau < 1 + 1/p$, $\delta = \frac{1-r}{2}$, $\sigma \in \mathbb{R}^+$, $u_0 \in W^{\tau,p}(\Omega)$, there is a unique maximal weak solution $u$ in $C(J; W^{1,p}(\Omega)) \cap C^\delta(J; W^{r,p}(\Omega)) \cap C(J \setminus \{0\}; W^{\sigma,p}(\Omega))$, with $J = [0,T]$, for some $T > 0$. No growth conditions are imposed to the coefficients. Related results are sketched in [8].

Replacing in the boundary condition the time derivative with the second term of the parabolic equation, one is formally reduced to a stationary boundary condition of second order, which is usually called "generalized Wentsell boundary condition". So, in [5] the authors consider the operator $\mathcal{A}u = \phi(x, u'(x))u''(x) + \psi(x, u(x), u'(x))$, with suitable assumptions on $\phi$ and $\psi$, in the interval $[0,1]$. General boundary conditions in the form $B(u)(j) := \alpha_j(Au)(j) + \beta_j u'(j) \in \gamma_j(u(j))$ ($j \in \{0,1\}$), with $\gamma_j$ maximal monotone are imposed. Then it is proved that $\mathcal{A}$, equipped with such conditions, is $m-$dissipative and generates a nonlinear contraction semigroup in $C([0,1])$.

In [10], the author considers a domain $\Omega$ with Lipschitz boundary, and the Laplacian with the Wentzell-Robin boundary condition

$$\Delta u + \frac{\partial u}{\partial \nu} + \beta u = 0.$$

Here $\beta \in L^\infty(\partial\Omega)$, $\beta \geq 0$. Then he proves that this operator generates an analytic semigroup in $C(\overline{\Omega})$.

The same author considers in [11] the $p-$Laplacian $A_p u = div(a(x)|\nabla u|^{p-2})\nabla u)$, with $a(x) > 0$ in $\Omega$ and the nonlinear Wentzell boundary condition on $\partial\Omega$

$$0 \in A_p u + b|\nabla u|^{p-2}\frac{\partial u}{\partial \nu} + \beta(\cdot, u),$$

Here $\beta(x,\cdot)$ is the sub differential of the functional $B(x,\cdot)$. He proves that it generates a nonlinear sub-Markovian $C_0-$semigroup on suitable $L^2-$spaces and, in case $b(x) \geq b_0 > 0$, he obtains the existence of nonlinear, non expansive semigroups in $L^q$ spaces, for every $q \in [1,\infty)$. A related situation with dynamic boundary conditions is treated in [6].

Finally, the asymptotic behavior of semilinear parabolic system with dynamic boundary conditions is studied in [1]. The nonlinearity is only in the boundary condition. A generalization (with a first order nonlinear term in the parabolic equation) is given in [2].

In this paper, differently from these papers, we want to show the existence and uniqueness of local solutions to (1.1) which are regular up to $t = 0$. Observe that we consider the case that the coefficients of the elliptic operator in the parabolic equation depend also on $\nabla u$. Moreover, we consider systems which are not necessarily in divergence form and no particular connection between the elliptic operator and the first order operator $\sum_{j=1}^n b_j(t,x',u)D_{x_j}$ is required: the only structural condition that we impose is that $\sum_{j=1}^n b_j(t,x',u)\nu_j(x') > 0$ for every $x' \in \partial\Omega$. Finally, we do not impose any kind of growth condition on the coefficients.

The organization of this paper is the following: in this first section, we introduce some notations, together with some known facts, which we are going to employ in the sequel. The second section contains a careful study of linear non autonomous parabolic systems, which may have some interest in itself, in particular Theorem 2.6. This study is preliminary to the final third section, containing the main result, Theorem 3.5. Such theorem states the existence and uniqueness of a local solution $u$ to (1.1) such that $u \in C^{1+\beta/2,2+\beta}((0,\tau) \times \Omega)$, with $D_t u_{|(0,\tau)\times\partial\Omega}$ bounded with values in $C^{1+\beta}(\partial\Omega)$

($\beta \in (0,1)$). Apart some regularity of the coefficients, we require only the strong ellipticity of the operator $\sum_{|\alpha|=2} a_\alpha(t,x,u,p)D_x^\alpha$ and the condition $\sum_{j=1}^n b_j(t,x',u)\nu_j(x') > 0$ if $x' \in \partial\Omega$. Concerning the initial datum $u_0$, it should belong to $C^{2+\beta}(\Omega)$ and a certain (necessary) compatibility condition ((3.4)) should hold.

The main tool of the proof is a certain maximal regularity result which was proved in [7], and it is stated in Theorem 2.1.

We pass to the aforementioned notations and known facts.

$C(\alpha, \beta, \dots)$ will indicate a positive real number depending on $\alpha, \beta, \dots$ and may be different from time to time. The symbol $\nabla_{x,u} b$ will indicate the gradient of $b$ with respect to the vector $(x,u)$ ($x \in \mathbb{R}^n, u \in \mathbb{R}$). On the other hand, $D^2_{x_j u} b$ stands for $\frac{\partial^2 b}{\partial x_j \partial u}$.

Let $\Omega$ be an open subset of $\mathbb{R}^n$. We shall indicate with $\mathcal{C}(\Omega)$ the class of complex valued continuous functions and with $C(\Omega)$ the subspace of uniformly continuous and bounded functions. If $f \in C(\Omega)$, it is continuously extensible to its topological closure $\overline{\Omega}$. We shall identify $f$ with this extension. If $m \in \mathbb{N}$, we indicate with $\mathcal{C}^m(\Omega)(C^m(\Omega))$ the class of functions $f$ in $\mathcal{C}(\Omega)(C(\Omega))$, whose derivatives $D^\alpha f$, with order $|\alpha| \le m$, belong to $\mathcal{C}(\Omega)(C(\Omega))$. $C^m(\Omega)$ admits the natural norm

$$\|f\|_{C^m(\Omega)} := \max\{\|D^\alpha f\|_{C(\Omega)} : |\alpha| \le m\}, \tag{1.4}$$

with $\|f\|_{C(\Omega)} := \sup_{x\in\Omega} |f(x)|$. If

$$[f]_{C^\beta(\Omega)} := \sup_{x,y\in\Omega, x\ne y} |x-y|^{-\beta}|f(x) - f(y)|\}$$

and $m \in \mathbb{N}_0$, we set

$$\|f\|_{C^{m+\beta}(\Omega)} := \max\{\|f\|_{C^m(\Omega)}, \max\{[D^\alpha f]_{C^\beta(\Omega)} : |\alpha| = m\}\}, \tag{1.5}$$

and, of course, $C^{m+\beta}(\Omega) = \{f \in C^m(\Omega) : \|f\|_{C^{m+\beta}(\Omega)} < \infty\}$. If $\partial\Omega$ is sufficiently regular, $0 \le \beta_0 < \beta < \beta_1$,

$$C^\beta(\Omega) \in J_{\frac{\beta-\beta_0}{\beta_1-\beta_0}}(C^{\beta_0}(\Omega), C^{\beta_1}(\Omega)),$$

which means that there exists $C > 0$, such that, $\forall f \in C^{\beta_1}(\Omega)$

$$\|f\|_{C^\beta(\Omega)} \le C\|f\|^{1-\theta}_{C^{\beta_0}(\Omega)}\|f\|^\theta_{C^{\beta_1}(\Omega)}. \tag{1.6}$$

We pass to consider vector valued functions. Let $X$ be a Banach space. If $A$ is a set, we shall indicate with $B(A;X)$ the Banach space of bounded functions from $A$ to $X$. If $m \in \mathbb{N}_0$, $\beta \ge 0$ and $\Omega$ is an open subset of $\mathbb{R}^n$, the definitions of $\mathcal{C}^m(\Omega;X)$, $C^m(\Omega;X)$, $C^\beta(\Omega;X)$ and of the norms $\|\cdot\|_{C^\beta(\Omega;X)}$ can be obtained by obvious modifications of the corresponding, in the case $X = \mathbb{C}$. (1.6) can be generalized to vector valued functions.

If $I$ is an open interval in $\mathbb{R}$, $\Omega$ is an open subset of $\mathbb{R}^n$ and $\alpha, \beta$ are nonnegative, we set

$$C^{\alpha,\beta}(I \times \Omega) := C^\alpha(I; C(\Omega)) \cap C^\beta(\Omega; C(I)), \tag{1.7}$$

equipped with its natural norm

$$\|f\|_{C^{\alpha,\beta}(I\times\Omega)} := \max\{\|f\|_{C^\alpha(I;C(\Omega))}, \|f\|_{B(I;C^\beta(\Omega))}\}.$$

The following inclusions hold (see [7]):

**Lemma 1.1.** *(I)* $C^\beta(\Omega; C(I)) \subseteq B(I; C^\beta(\Omega))$.

*(II) Suppose* $\alpha, \beta \ge 0$ *with* $\beta \notin \mathbb{Z}$ *and* $\Omega$ *such that there exists a common linear bounded extension operator, mapping* $C(\Omega)$ *into* $C(\mathbb{R}^n)$ *and* $C^\beta(\Omega)$ *into* $C^\beta(\mathbb{R}^n)$. *Then,*

$$C^{\alpha,\beta}(I \times \Omega) = C^\alpha(I; C(\Omega)) \cap B(I; C^\beta(\Omega)).$$

*Let* $\beta \in (0,1)$ *and suppose that there exists a common linear bounded extension operator mapping* $C^\gamma(\Omega)$ *into* $C^\gamma(\mathbb{R}^n)$, $\forall \gamma \in [0, 2+\beta]$. *Then*

*(III)* $C^{1+\beta/2,2+\beta}(I \times \Omega) = C^{1+\beta/2}(I; C(\Omega)) \cap B(I; C^{2+\beta}(\Omega))$;

*(IV) if* $f \in C^{1+\beta/2,2+\beta}(I \times \Omega)$, $D_t f \in B(I; C^\beta(\Omega))$;

*(V)* $C^{1+\beta/2,2+\beta}(I \times \Omega) \subseteq C^{\frac{1+\beta}{2}}(I; C^1(\Omega)) \cap C^{\beta/2}(I; C^2(\Omega))$.

We shall need also spaces $C^{\alpha,\beta}(I \times V)$, with $V$ suitably regular submanifold of $\mathbb{R}^n$: we shall consider, in particular, the case $V = \partial\Omega$, with $\Omega$ open, bounded subset of $\mathbb{R}^n$. Of course, in this case $C^\beta(V; C(I))$ can be defined by local charts.

We shall employ the following version of the continuation method:

**Proposition 1.2.** *Let $X, Y$ be Banach spaces and $L \in C([0,1]; \mathcal{L}(X,Y))$. Assume the following:*
*(a) there exists $M \in \mathbb{R}^+$, such that, $\forall x \in X$, $\forall \epsilon \in [0,1]$,*

$$\|x\|_X \leq M \|L(\epsilon)x\|_Y;$$

*(b) $L(0)$ is onto $Y$.*
*Then, $\forall \epsilon \in [0,1]$ $L(\epsilon)$ is a linear and topological isomorphism between $X$ and $Y$.*

# 2 Nonautonomous linear systems

In this section we shall consider the following system

$$\begin{cases} D_t u(t,x) - A(t,x,D_x)u(t,x) = f(t,x), \quad t \in (0,T), x \in \Omega, \\ \\ D_t u(t,x') + B(t,x',D_x)u(t,x') = h(t,x'), \quad t \in (0,T), x' \in \partial\Omega, \\ \\ u(0,x) = u_0(x), \quad x \in \Omega, \end{cases} \tag{2.1}$$

with the following conditions:

*(AF1) $\Omega$ is an open bounded subset of $\mathbb{R}^n$, lying on one side of its boundary $\partial\Omega$, which is a submanifold of class $C^{2+\beta}$ of $\mathbb{R}^n$, for some $\beta \in (0,1)$;*

*(AF2) $A(t,x,D_x) = \sum_{|\alpha|\leq 2} a_\alpha(t,x) D_x^\alpha$, $\sum_{|\alpha|\leq 2} \|a_\alpha\|_{C^{\beta/2,\beta}((0,T)\times\Omega)} \leq N$; if $|\alpha| = 2$, $a_\alpha$ is real valued and $\sum_{|\alpha|=2} a_\alpha(t,x)\eta^\alpha \geq \nu|\eta|^2$, for some $N, \nu \in \mathbb{R}^+$, $\forall (t,x) \in [0,T] \times \overline{\Omega}$, $\forall \eta \in \mathbb{R}^n$;*

*(AF3) $B(t,x',D_x) = \sum_{|\alpha|\leq 1} b_\alpha(t,x) D_x^\alpha$, $\sum_{|\alpha|\leq 1} \|b_\alpha\|_{C^{\beta/2,1+\beta}((0,T)\times\partial\Omega)} \leq N$; if $|\alpha| = 1$, $b_\alpha$ is real valued and $\sum_{|\alpha|=1} b_\alpha(t,x')\nu(x')^\alpha \geq \nu$ $\forall x' \in \partial\Omega$.*

In order to study (2.1), we consider the autonomous system

$$\begin{cases} D_t u(t,x) - A(x,D_x)u(t,x) = f(t,x), \quad t \in (0,T), x \in \Omega, \\ \\ D_t u(t,x') + B(x',D_x)u(t,x') = h(t,x'), \quad t \in (0,T), x' \in \partial\Omega, \\ \\ u(0,x) = u_0(x), \quad x \in \Omega, \end{cases} \tag{2.2}$$

We recall the following result, which was proved in [7]:

**Theorem 2.1.** *Consider system (2.2), with the following conditions (AE1)-(AE3):*

*(AE1) (AF1) holds;*

*(AE2) $a_\alpha \in C^\beta(\Omega)$, $\forall \alpha \in \mathbb{N}_0^n$, $|\alpha| \leq 2$; if $|\alpha| = 2$, $a_\alpha$ is real valued and $\sum_{|\alpha|=2} a_\alpha(x)\eta^\alpha \geq \nu|\eta|^2$, for some $\nu \in \mathbb{R}^+$, $\forall x \in \overline{\Omega}$, $\forall \eta \in \mathbb{R}^n$;*

*(AE3) $B(x',D_x) = \sum_{|\alpha|\leq 1} b_\alpha(x) D_x^\alpha$, $b_\alpha \in C^{1+\beta}(\partial\Omega)$ $\forall \alpha \in \mathbb{N}_0^n$, $|\alpha| \leq 1$; if $|\alpha| = 1$, $b_\alpha$ is real valued and $\sum_{|\alpha|=1} b_\alpha(x')\nu(x')^\alpha > 0$ $\forall x' \in \partial\Omega$.*

*Then the following conditions are necessary and sufficient, in order that (2.2) have a unique solution $u$ in $C^{1+\beta/2,2+\beta}((0,T)\times\Omega)$, with $D_t u_{|(0,T)\times\partial\Omega)}$ in $B((0,T); C^{1+\beta}(\partial\Omega))$:*
*(a) $f \in C^{\beta/2,\beta}((0,T)\times\Omega))$;*
*(b) $h \in C^{\beta/2,1+\beta}((0,T)\times\partial\Omega)$;*
*(c) $u_0 \in C^{2+\beta}(\Omega)$;*
*(d) $\forall \xi' \in \partial\Omega$*

$$A(\xi',D_\xi)u_0(\xi') + f(0,\xi') = -B(\xi',D_\xi)u_0(\xi') + h(0,\xi').$$

**Lemma 2.2.** *Assume that (AE1)-(AE3), $T_0 \in \mathbb{R}^+$ and $T \leq T_0$. Suppose that $f$, $h$, $u_0$ satisfy conditions (a)-(d) in the statement of Corollary* **??**. *Let $u$ be the solution in $C^{1+\beta/2,2+\beta}((0,T)\times\Omega)$, with $D_t u \in B((0,T); C^{1+\beta}(\partial\Omega))$ of (2.2). Then:*
*(I) there exists $C(T_0,A,B)$ in $\mathbb{R}^+$, such that*

$$\|u\|_{C^{1+\beta/2,2+\beta}((0,T)\times\Omega))} + \|u\|_{C^{\beta/2}((0,T);C^2(\Omega))} + \|u\|_{C^{\frac{1+\beta}{2}}((0,T);C^1(\Omega))}$$

$$\leq C(T_0,A,B)(\|f\|_{C^{\beta/2,\beta}((0,T)\times\Omega)} + \|u_0\|_{C^{2+\beta}(\Omega)} + \|h\|_{C^{\beta/2,1+\beta}((0,T)\times\partial\Omega)}).$$

*(II) Suppose that $u_0 = 0$. Then, if $0 \le \theta \le 1$,*

$$\|u\|_{C^{\theta}((0,T);C(\Omega))} \le C(T_0, A, B)T^{1-\theta}(\|f\|_{C^{\beta/2,\beta}((0,T)\times\Omega)} + \|h\|_{C^{\beta/2,1+\beta}((0,T)\times\partial\Omega)}), \tag{2.3}$$

*if $0 \le \theta \le 2+\beta$,*

$$\|u\|_{B((0,T);C^{\theta}(\Omega))} \le C(T_0, A, B, \theta)T^{\frac{2+\beta-\theta}{2+\beta}}(\|f\|_{C^{\beta/2,\beta}((0,T)\times\Omega)} + \|h\|_{C^{\beta/2,1+\beta}((0,T)\times\partial\Omega)}). \tag{2.4}$$

*(III) Again supposing $u_0 = 0$, we have*

$$\|u\|_{C^{\beta/2}((0,T);C^1(\Omega))} \le C(T_0, A, B)T^{1/2}(\|f\|_{C^{\beta/2,\beta}((0,T)\times\Omega)} + \|h\|_{C^{\beta/2,1+\beta}((0,T)\times\partial\Omega)}).$$

*Proof* We extend $f$ and $h$ to elements $\tilde{f}$ and $\tilde{h}$ in (respectively) $C^{\beta/2,\beta}((0,T_0)\times\Omega)$ and $C^{\beta/2,1+\beta}((0,T_0)\times\partial\Omega)$: we set

$$\tilde{f}(t,x) = \begin{cases} f(t,\xi) & \text{if } 0 \le t \le T, \xi \in \Omega, \\ f(T,\xi) & \text{if } 0 \le T \le t \le T_0, \xi \in \Omega, \end{cases}$$

$$\tilde{h}(t,\xi') = \begin{cases} h(t,\xi') & \text{if } 0 \le t \le T, \xi' \in \partial\Omega, \\ h(T,\xi') & \text{if } 0 \le T \le t \le T_0, \xi' \in \partial\Omega. \end{cases}$$

We denote with $\tilde{u}$ the solution to

$$\begin{cases} D_t\tilde{u}(t,\xi) = A(\xi, D_\xi)\tilde{u}(t,\xi) + \tilde{f}(t,\xi), \quad t \in [0,T_0], \xi \in \Omega, \\ \tilde{u}(0,\xi) = u_0(\xi), \quad \xi \in \Omega, \\ D_t\tilde{u}(t,\xi') + B(\xi', D_\xi)\tilde{u}(t,\xi') = \tilde{h}(t,\xi'), \quad t \in [0,T_0], \xi' \in \partial\Omega. \end{cases}$$

Clearly, $\tilde{u}$ is an extension of $u$. So

$$\begin{aligned} &\|u\|_{C^{1+\beta/2,2+\beta}((0,T)\times\Omega)} + \|D_t u_{|(0,T)\times\partial\Omega}\|_{B((0,T);C^{1+\beta}(\partial\Omega))} + \|u\|_{C^{\beta/2}((0,T);C^2(\Omega))} \\ &\le \|\tilde{u}\|_{C^{1+\beta/2,2+\beta}((0,T_0)\times\Omega)} + \|D_t\tilde{u}_{|(0,T)\times\partial\Omega}\|_{B((0,T_0);C^{1+\beta}(\partial\Omega))} + \|\tilde{u}\|_{C^{\beta/2}((0,T_0);C^2(\Omega))} \\ &\le C(T_0, A, B)(\|\tilde{f}\|_{C^{\beta/2,\beta}((0,T_0)\times\Omega)} + \|u_0\|_{C^{2+\beta}(\Omega)} + \|\tilde{h}\|_{C^{\beta/2,1+\beta}((0,T_0)\times\partial\Omega)}) \\ &= C(T_0, A, B)(\|f\|_{C^{\beta/2,\beta}((0,T)\times\Omega)} + \|u_0\|_{C^{2+\beta}(\Omega)} + \|h\|_{C^{\beta/2,1+\beta}((0,T)\times\partial\Omega)}). \end{aligned}$$

So (I) is proved.

Concerning (II), we begin by considering the case $\theta = 0$. If $0 \le t \le T$, we have

$$u(t,\cdot) = \int_0^t D_s u(s,\cdot)ds,$$

so that

$$\begin{aligned} \|u(t,\cdot)\|_{C(\Omega)} &\le \textstyle\int_0^t \|D_s u(s,\cdot)\|_{C(\Omega)}ds \le t\|u\|_{C^{1+\beta/2}((0,T);C(\Omega))} \\ &\le C(T_0, A, B)T(\|f\|_{C^{\beta/2,\beta}((0,T)\times\Omega)} + \|h\|_{C^{\beta/2,1+\beta}((0,T)\times\partial\Omega)}), \end{aligned}$$

employing (I). The cases $0 < \theta \le 1$ follow from the foregoing from $\theta = 0$ and the fact that $C^{\theta}((0,T);C(\Omega)) \in J_{\theta}(C((0,T);C(\Omega)), C^1((0,T);C(\Omega))$. So (2.6) is proved. Finally (2.7) follows from (2.6) with $\theta = 0$, (I) and the fact that $C^{\theta}(\Omega)) \in J_{\theta/(2+\beta)}(C(\Omega), C^{2+\beta}(\Omega))$.

It remains to consider (III). We have

$$\begin{aligned} \|u\|_{C^{\beta/2}((0,T);C^1(\Omega))} &= \|u\|_{C((0,T);C^1(\Omega))} + [u]_{C^{\beta/2}((0,T);C^1(\Omega))} \\ &\le \|u\|_{C((0,T);C^1(\Omega))} + T^{1/2}[u]_{C^{(1+\beta)/2}((0,T);C^1(\Omega))}, \end{aligned}$$

and the conclusion follows from (I) and (II).

□

**Lemma 2.3.** *We consider a system in the form*

$$\begin{cases} D_t u(t,\xi) - A(\xi, D_\xi)u(t,\xi) - \mathcal{A}(t)u(t,\cdot)(\xi) = f(t,\xi), \quad t \in (0,T), \xi \in \Omega, \\ \\ D_t u(t,\xi') + B(\xi', D_\xi)u(t,\xi') + \mathcal{B}(t)u(t,\cdot)(\xi') = h(t,\xi'), \quad t \in (0,T), \xi' \in \partial\Omega, \\ \\ u(0,\xi) = u_0(\xi), \quad \xi \in \Omega, \end{cases} \tag{2.5}$$

*with the following conditions:*

*(a) (AE1)-(AE3) are satisfied;*

*(b)* $\forall t \in [0,T]$ $\mathcal{A}(t) \in \mathcal{L}(C^2(\Omega), C(\Omega)) \cap \mathcal{L}(C^{2+\beta}(\Omega), C^\beta(\Omega))$ *and, for certain* $\delta$ *and* $M$ *in* $\mathbb{R}^+$, $\forall t \in [0,T]$, $\forall u \in C^{2+\beta}(\Omega))$,

$$\|\mathcal{A}(t)u\|_{C(\Omega))} \leq \delta \|u\|_{C^2(\Omega)} + M\|u\|_{C^1(\Omega)},$$

$$\|\mathcal{A}\|_{C^{\beta/2}((0,T);\mathcal{L}(C^2(\Omega),C(\Omega)))} \leq M,$$

$$\|\mathcal{A}(t)u\|_{C^\beta(\Omega)} \leq \delta\|u\|_{C^{2+\beta}(\Omega)} + M\|u\|_{C^2(\Omega)};$$

*(c)* $\forall t \in [0,T]$ $\mathcal{B}(t) \in \mathcal{L}(C^1(\Omega), C(\partial\Omega)) \cap \mathcal{L}(C^{2+\beta}(\Omega), C^{1+\beta}(\partial\Omega))$ *and,* $\forall t \in [0,T]$, $\forall u \in C^{2+\beta}(\Omega)$,

$$\|\mathcal{B}\|_{C^{\beta/2}((0,T);\mathcal{L}(C^1(\Omega),C(\partial\Omega)))} \leq M,$$

$$\|\mathcal{B}(t)u\|_{C^{1+\beta}(\partial\Omega)} \leq \delta\|u\|_{C^{2+\beta}(\Omega)} + M\|u\|_{C^2(\Omega)}.$$

*Consider the system (2.5). Then:*

*(I) there exists* $\delta_0 \in \mathbb{R}^+$*, depending only on* $A$ *and* $B$*, such that, if* $\delta \leq \delta_0$, $f \in C^{\beta/2,\beta}((0,T)\times\Omega)$, $h \in C^{\beta/2,1+\beta}((0,T)\times\partial\Omega)$, $u_0 \in C^{2+\beta}(\Omega)$ *and*

$$A(\xi', D_\xi)u_0(\xi') + \mathcal{A}(0)u_0(\xi') + f(0,\xi') = -B(\xi', D_\xi)u_0(\xi') - \mathcal{B}(0)u_0(\xi') + h(0,\xi') \quad \forall \xi' \in \partial\Omega,$$

*(2.5) has a unique solution* $u$ *in* $C^{1+\beta/2,2+\beta}((0,T)\times\Omega)$*, with* $D_t u_{|(0,T)\times\partial\Omega} \in B((0,T); C^{1+\beta}(\partial\Omega))$.

*Let* $T_0 \in \mathbb{R}^+$*. Assume that (b) and (c) are satisfied and (I) holds, replacing* $T$ *with* $T_0$*. Take* $0 < T \leq T_0$*. Then:*

*(II) there exists* $C(T_0, A, B, \delta, M)$ *in* $\mathbb{R}^+$*, such that*

$$\|u\|_{C^{1+\beta/2,2+\beta}((0,T)\times\Omega))} + \|D_t u_{|(0,T)\times\partial\Omega}\|_{B((0,T);C^{1+\beta}(\partial\Omega))} + \|u\|_{C^{\beta/2}((0,T);C^2(\Omega))} + \|u\|_{C^{\frac{1+\beta}{2}}((0,T);C^1(\Omega))}$$

$$\leq C(T_0, A, B, \delta, M)(\|f\|_{C^{\beta/2,\beta}((0,T)\times\Omega)} + \|u_0\|_{C^{2+\beta}(\Omega)} + \|h\|_{C^{\beta/2,1+\beta}((0,T)\times\partial\Omega)}).$$

*(III) Suppose that* $u_0 = 0$*. Then, if* $0 \leq \theta \leq 1$,

$$\|u\|_{C^\theta((0,T);C(\Omega))} \leq C(T_0, A, B, \delta, M)T^{1-\theta}(\|f\|_{C^{\beta/2,\beta}((0,T)\times\Omega)} + \|h\|_{C^{\beta/2,1+\beta}((0,T)\times\partial\Omega)}), \tag{2.6}$$

*if* $0 \leq \theta \leq 2+\beta$,

$$\|u\|_{B((0,T);C^\theta(\Omega))} \leq C(T_0, A, B, \delta, M, \theta)T^{\frac{2+\beta-\theta}{2+\beta}}(\|f\|_{C^{\beta/2,\beta}((0,T)\times\Omega)} + \|h\|_{C^{\beta/2,1+\beta}((0,T)\times\partial\Omega)}). \tag{2.7}$$

*(IV) Again supposing* $u_0 = 0$*, we have*

$$\|u\|_{C^{\beta/2}((0,T);C^1(\Omega))} \leq C(T_0, A, B, \delta, M)T^{1/2}(\|f\|_{C^{\beta/2,\beta}((0,T)\times\Omega)} + \|h\|_{C^{\beta/2,1+\beta}((0,T)\times\partial\Omega)}).$$

*Proof* We prove the result in several steps.

*Step 1. We consider the case* $u_0 = 0$ *and prove an a priori estimate if* $T$ *is sufficiently small.*

So we have

$$f(0,\xi') = h(0,\xi') \quad \forall \xi' \in \partial\Omega,$$

and we suppose that $u \in C^{1+\beta/2,2+\beta}((0,T)\times\Omega)$, with $D_t u_{|(0,T)\times\partial\Omega} \in B((0,T); C^{1+\beta}(\partial\Omega))$, solves (2.5), with $u_0 = 0$. It is easily seen that, if we set

$$F(t,\xi) := \mathcal{A}(t)u(t,\cdot)(\xi) + f(t,\xi), \quad (t,\xi) \in (0,T)\times\Omega,$$

and

$$H(t,\xi') := -\mathcal{B}(t)u(t,\cdot)(\xi') + h(t,\xi'), \quad (t,\xi') \in (0,T)\times\partial\Omega,$$

we have that $F \in C^{\beta/2,\beta}((0,T)\times\Omega)$ and $H \in C^{\beta/2,1+\beta}((0,T)\times\partial\Omega)$. So, by Lemma 2.2, if (say) $T \leq 1$, we have

$$\begin{aligned}
\|u\|_{C^{1+\beta/2,2+\beta}((0,T)\times\Omega))} + \|D_t u_{|(0,T)\times\partial\Omega}\|_{B((0,T);C^{1+\beta}(\partial\Omega))} + \|u\|_{C^{\beta/2}((0,T);C^2(\Omega))} + \|u\|_{C^{\frac{1+\beta}{2}}((0,T);C^1(\Omega))} \\
+T^{-\frac{\beta}{2+\beta}}\|u\|_{B((0,T);C^2(\Omega))} + T^{-1/2}\|u\|_{C^{\beta/2}((0,T);C^1(\Omega))} \\
\leq C(A,B)(\|f\|_{C^{\beta/2,\beta}((0,T)\times\Omega))} + \|h\|_{C^{\beta/2,1+\beta}((0,T)\times\partial\Omega))} \\
+\|\mathcal{A}u\|_{C^{\beta/2,\beta}((0,T)\times\Omega))} + \|\mathcal{B}u\|_{C^{\beta/2,1+\beta}((0,T)\times\partial\Omega))}).
\end{aligned} \tag{2.8}$$

If $0 \leq s < t \leq T$, we have

$$\begin{aligned}
\|\mathcal{A}(t)u(t) - \mathcal{A}(s)u(s)\|_{C(\Omega))} \\
\leq \|[\mathcal{A}(t)-\mathcal{A}(s)]u(t)\|_{C(\Omega))} + \|\mathcal{A}(s)(u(t)-u(s))\|_{C(\Omega))} \\
\leq M(t-s)^{\beta/2}\|u(t)\|_{C^2(\Omega)} + \delta\|u(t)-u(s)\|_{C^2(\Omega))} + M\|u(t)-u(s)\|_{C^1(\Omega))}
\end{aligned}$$

so that

$$[\mathcal{A}u]_{C^{\beta/2}((0,T);C(\Omega))} \leq M\|u\|_{B((0,T);C^2(\Omega))} + \delta\|u\|_{C^{\beta/2}((0,T);C^2(\Omega))} + M\|u\|_{C^{\beta/2}((0,T);C^1(\Omega))}$$

and

$$\begin{aligned}
\|\mathcal{A}u\|_{C^{\beta/2,\beta}((0,T)\times\Omega))} = \|\mathcal{A}u\|_{B((0,T);C^\beta(\Omega))} + [\mathcal{A}u]_{C^{\beta/2}((0,T);C(\Omega))} \\
\leq \delta(\|u\|_{B((0,T);C^{2+\beta}(\Omega)} + \|u\|_{C^{\beta/2}((0,T);C^2(\Omega))}) + 2M\|u\|_{B((0,T);C^2(\Omega))} + M\|u\|_{C^{\beta/2}((0,T);C^1(\Omega))}.
\end{aligned} \tag{2.9}$$

Analogously, we can deduce, from (c),

$$\begin{aligned}
\|\mathcal{B}u\|_{C^{\beta/2,1+\beta}((0,T)\times\partial\Omega))} = \|\mathcal{B}u\|_{B((0,T);C^{1+\beta}(\partial\Omega))} + [\mathcal{B}u]_{C^{\beta/2}((0,T);C(\partial\Omega))} \\
\leq \delta\|u\|_{B((0,T);C^{2+\beta}(\Omega))} + M(\|u\|_{B((0,T);C^2(\Omega))} + \|u\|_{C^{\beta/2}((0,T);C^1(\Omega))})
\end{aligned} \tag{2.10}$$

and, from (2.8)-(2.10),

$$\begin{aligned}
\|u\|_{C^{1+\beta/2,2+\beta}((0,T)\times\Omega))} + \|D_t u_{|(0,T)\times\partial\Omega}\|_{B((0,T);C^{1+\beta}(\partial\Omega))} + \|u\|_{C^{\beta/2}((0,T);C^2(\Omega))} \\
+T^{-\frac{\beta}{2+\beta}}\|u\|_{B((0,T);C^2(\Omega))} + T^{-1/2}\|u\|_{C^{\beta/2}((0,T);C^1(\Omega))} \\
\leq C(A,B)[\|f\|_{C^{\beta/2,\beta}((0,T)\times\Omega))} + \|h\|_{C^{\beta/2,1+\beta}((0,T)\times\partial\Omega))} \\
+2\delta(\|u\|_{B((0,T);C^{2+\beta}(\Omega))} + \|u\|_{C^{\beta/2}((0,T);C^2(\Omega))}) + 3M\|u\|_{B((0,T);C^2(\Omega))} \\
+2M\|u\|_{C^{\beta/2}((0,T);C^1(\Omega))}].
\end{aligned}$$

Now we take $\delta \leq \delta_0$ in $\mathbb{R}^+$ and $T$ so small that

$$2C(A,B)\delta_0 \leq \frac{1}{2}, \quad 3MC(A,B) \leq \frac{1}{2}T^{-\frac{\beta}{2+\beta}}, \quad 2MC(A,B) \leq \frac{1}{2}T^{-1/2}. \tag{2.11}$$

We deduce the a priori estimate

$$\begin{aligned}
\|u\|_{C^{1+\beta/2,2+\beta}((0,T)\times\Omega))} + \|D_t u_{|(0,T)\times\partial\Omega}\|_{B((0,T);C^{1+\beta}(\partial\Omega))} + \|u\|_{C^{\beta/2}((0,T);C^2(\Omega))} + \|u\|_{C^{\frac{1+\beta}{2}}((0,T);C^1(\Omega))} \\
+T^{-\frac{\beta}{2+\beta}}\|u\|_{B((0,T);C^2(\Omega))} + T^{-1/2}\|u\|_{C^{\beta/2}((0,T);C^1(\Omega))} \\
\leq 2C(A,B)[\|f\|_{C^{\beta/2,\beta}((0,T)\times\Omega))} + \|h\|_{C^{\beta/2,1+\beta}((0,T)\times\partial\Omega))}].
\end{aligned} \tag{2.12}$$

*Step 2. We show the existence of a unique solution if $u_0 = 0$ and (2.11) holds.*

To this aim, we consider, for every $\rho \in [0,1]$, the system

$$\begin{cases} D_t u(t,\xi) - A(\xi, D_\xi)u(t,\xi) - \rho\mathcal{A}(t)u(t,\cdot)(\xi) = f(t,\xi), & t \in (0,T), \xi \in \Omega, \\ D_t u(t,\xi') + B(\xi', D_\xi)u(t,\xi') + \rho\mathcal{B}(t)u(t,\cdot)(\xi') = h(t,\xi'), & t \in (0,T), \xi' \in \partial\Omega, \\ u(0,\xi) = u_0(\xi), \quad \xi \in \Omega, \end{cases} \tag{2.13}$$

We set

$$X := \{u \in C^{1+\beta/2,2+\beta}((0,T)\times\Omega) : D_t u_{|(0,T)\times\partial\Omega} \in B((0,T); C^{1+\beta}(\partial\Omega)), u(0,\cdot) = 0\},$$

$$Y := \{(f,h) \in C^{\beta/2,\beta}((0,T)\times\Omega) \times C^{\beta/2,1+\beta}((0,T)\times\partial\Omega) : f(0,\cdot)_{|\partial\Omega} = h(0,\cdot)\}.$$

$X$ and $Y$ are Banach spaces with natural norms. For each $\rho$ we define the following operator $T_\rho$:

$$\begin{cases} T_\rho : X \to Y, \\ T_\rho u = (D_t u - A(\xi, D_\xi)u - \rho\mathcal{A}u, D_t u_{|(0,T)\times\partial\Omega} + B(\xi', D_\xi)u + \rho\mathcal{B}u). \end{cases}$$

Then $\rho \to T_\rho$ belongs to $C([0,1]; \mathcal{L}(X,Y))$ and $T_0$ is a linear and topological isomorphism between $X$ and $Y$ by Theorem 2.1. Moreover, it is clear that for every $\rho \in [0,1]$, (b) and (c) are satisfied if we replace $\mathcal{A}(t)$ with $\rho\mathcal{A}(t)$ and $\mathcal{B}(t)$ with $\rho\mathcal{B}(t)$ with the same constants $\delta$ and $M$. So, by the a priori estimate (2.12) and the continuation principle, we deduce Step 2.

*Step 3. We prove (I), continuing to assume $\delta \le \delta_0$ and $\delta_0$ and $T$ satisfying (2.11).*

We take $v(t,\xi) := u(t,\xi) - u_0(\xi)$ as new unknown. We obtain the system

$$\begin{cases} D_t v(t,\xi) - A(\xi, D_\xi)v(t,\xi) - \mathcal{A}(t)v(t,\cdot)(\xi) = A(\xi, D_\xi)u_0(\xi) + \mathcal{A}(t)u_0(\xi) + f(t,\xi), \quad t \in (0,T), \xi \in \Omega, \\ D_t v(t,\xi') + B(\xi', D_\xi)v(t,\xi') + \mathcal{B}(t)v(t,\cdot)(\xi') = -B(\xi', D_\xi)u_0(\xi') - \mathcal{B}(t)u_0(\xi') + h(t,\xi'), \\ t \in (0,T), \quad \xi' \in \partial\Omega, \\ v(0,\xi) = 0, \quad \xi \in \Omega, \end{cases} \tag{2.14}$$

to which Step 2 is applicable.

*Step 4. Proof of (I).*

We begin by showing the uniqueness. So we suppose that $u \in C^{1+\beta/2,2+\beta}((0,T)\times\Omega)$, $D_t u_{|(0,T)\times\partial\Omega} \in B((0,T); C^{1+\beta}(\partial\Omega))$ and

$$\begin{cases} D_t u(t,\xi) - A(\xi, D_\xi)u(t,\xi) - \mathcal{A}(t)u(t,\cdot)(\xi) = 0, & t \in (0,T), \xi \in \Omega, \\ D_t u(t,\xi') + B(\xi', D_\xi)u(t,\xi') + \mathcal{B}(t)u(t,\cdot)(\xi') = 0, & t \in (0,T), \xi' \in \partial\Omega, \\ u(0,\xi) = 0, \quad \xi \in \Omega, \end{cases}$$

We suppose, by contradiction, that $u \ne 0$. Let $\tau := \inf\{t \in [0,T] : u(t,\cdot) \ne 0\}$. Then $0 \le \tau < T$ and $u(\tau,\cdot) = 0$. We set

$$u_1(t,\cdot) := u(\tau + t, \cdot), \quad t \in [0, T-\tau].$$

Then, if $0 < T_1 \le T - \tau$, $u_1$ solves the system

$$\begin{cases} D_t u_1(t,\xi) - A(\xi, D_\xi)u_1(t,\xi) - \mathcal{A}(\tau+t)u_1(t,\cdot)(\xi) = 0, & t \in (0,T_1), \xi \in \Omega, \\ D_t u_1(t,\xi') + B(\xi', D_\xi)u_1(t,\xi') + \mathcal{B}(\tau+t)u_1(t,\cdot)(\xi') = 0, & t \in (0,T_1), \xi' \in \partial\Omega, \\ u_1(0,\xi) = 0, \quad \xi \in \Omega. \end{cases}$$

Clearly, the families of operators $\{\mathcal{A}(\tau+t) : t \in [0, T-\tau]\}$ and $\{\mathcal{B}(\tau+t) : t \in [0, T-\tau]\}$ satisfy the conditions (b) and (c) with the same constants $\delta$ and $M$. So we deduce from Step 4 that, if we take $T_1$ sufficiently small, $u_{1|(0,T_1)\times\Omega} = 0$, so that $u_{|(0,\tau+T_1)\times\Omega} = 0$, in contradiction with the definition of $\tau$.

We show the existence. We suppose that $T$ does not satisfy one of the majorities in (2.11) and replace it with $\tau \in (0, T)$, in such a way that these majorities are satisfied by $\tau$. So we can apply Step 3 and get a unique solution $u_1$ with domain $[0, \tau] \times \overline{\Omega}$. We observe that

$$\begin{aligned} &A(\xi', D_\xi)u_1(\tau, \xi') + \mathcal{A}(\tau)u_1(\tau, \cdot)(\xi') + f(\tau, \xi') = D_t u_1(\tau, \xi') \\ &= -B(\xi', D_\xi)u_1(\tau, \xi') - \mathcal{B}(\tau)u_1(\tau, \cdot)(\xi') + h(\tau, \xi') \quad \forall \xi' \in \partial\Omega, \end{aligned} \tag{2.15}$$

and consider the system

$$\begin{cases} D_t u_2(t, \xi) - A(\xi, D_\xi)u_2(t, \xi) - \mathcal{A}(\tau + t)u_2(t, \cdot)(\xi) = f(\tau + t, \xi), \quad t \in (0, \tau \wedge (T - \tau)), \xi \in \Omega, \\ D_t u_2(t, \xi') + B(\xi', D_\xi)u_2(t, \xi') + \mathcal{B}(\tau + t)u_2(t, \cdot)(\xi') = h(\tau + t, \xi'), \quad t \in (0, \tau \wedge (T - \tau)), \xi' \in \partial\Omega, \\ u_2(0, \xi) = u_1(\tau, \xi), \quad \xi \in \Omega, \end{cases}$$

By (2.15) Step 3 is applicable and it is easily seen that, if we set

$$u(t, \xi) = \begin{cases} u_1(t, \xi) & \text{if } (t, \xi) \in [0, \tau] \times \overline{\Omega}, \\ u_2(t - \tau, \xi) & \text{if } (t, \xi) \in [\tau, 2\tau \wedge T] \times \overline{\Omega}, \end{cases}$$

$u \in C^{1+\beta/2, 2+\beta}((0, (2\tau) \wedge T) \times \Omega)$, with $D_t u_{|(0,(2\tau)\wedge T) \times \partial\Omega} \in B((0, (2\tau) \wedge T); C^{1+\beta}(\partial\Omega))$ and solves (2.5) if we replace $T$ with $(2\tau) \wedge T$. In case $2\tau < T$, we can iterate the procedure and in a finite number of steps we construct a solution in $(0, T) \times \Omega$.

II) If $T$ is so small that (2.11) holds and $u_0 = 0$, we get the conclusion from the a priori estimate (2.12). The case $u_0 \neq 0$, can be deduced applying the foregoing to the solution $v$ to (2.14). If $T$ does not satisfy (2.11), we fix $T_1$ in $(0, T)$, satisfying it and obtain (II) applying the estimates in an interval of length, less or equal than $T_1$ $[T/T_1] + 1$ times.

(III)-(IV) can be proved with the same arguments employed for the analogous estimates in Lemma 2.2.

□

**Remark 2.4.** Lemma 2.2 is applicable in case

$$\mathcal{A}(t)u(\xi) = \sum_{|\alpha| \leq 2} r_\alpha(t, \xi) D_\xi^\alpha u(\xi), \quad (t, \xi) \in [0, T] \times \overline{\Omega}, \tag{2.16}$$

and

$$\mathcal{B}(t)u(\xi') = \sum_{|\alpha| \leq 1} \sigma_\alpha(t, \xi') D_x^\alpha u(\xi'), \quad (t, \xi) \in [0, T] \times \partial\Omega, \tag{2.17}$$

with $r_\alpha \in C^{\beta/2,\beta}((0, T) \times \Omega)$ ($|\alpha| \leq 2$) and $\sigma_\alpha \in C^{\beta/2,1+\beta}((0, T) \times \partial\Omega)$ ($|\alpha| \leq 1$), if

$$\sum_{|\alpha|=2} \|r_\alpha\|_{C((0,T)\times\Omega)} + \sum_{|\alpha|=1} \|\sigma_\alpha\|_{C((0,T)\times\partial\Omega)} \leq d \tag{2.18}$$

and

$$\sum_{|\alpha| \leq 2} \|r_\alpha\|_{C^{\beta/2,\beta}((0,T)\times\Omega)} + \sum_{|\alpha| \leq 1} \|\sigma_\alpha\|_{C^{\beta/2,1+\beta}((0,T)\times\partial\Omega)} \leq N, \tag{2.19}$$

with $d$ and $N$ in $\mathbb{R}^+$. Then we have

$$\|\mathcal{A}(t)u\|_{C(\Omega)} \leq C(n)(d\|u\|_{C^2(\Omega)} + N\|u\|_{C^1(\Omega)}), \tag{2.20}$$

$$\|\mathcal{A}\|_{C^{\beta/2}((0,T);\mathcal{L}(C^2(\Omega), C(\Omega)))} \leq C(n)N, \tag{2.21}$$

$$\|\mathcal{A}(t)u\|_{C^\beta(\Omega)} \leq C(n)(\delta\|u\|_{C^{2+\beta}(\Omega)} + N\|u\|_{C^2(\Omega)}). \tag{2.22}$$

Moreover,

$$\|\mathcal{B}\|_{C^{\beta/2}((0,T);\mathcal{L}(C^1(\Omega), C(\partial\Omega)))} \leq C(n)N,$$

$$\|\mathcal{B}(t)u\|_{C^{1+\beta}(\partial\Omega)} \leq C(n)(\delta\|u\|_{C^{2+\beta}(\Omega)} + N\|u\|_{C^2(\Omega)}).$$

So we can take $\delta = C(n)d$ and $M = C(n)N$.

**Lemma 2.5.** *Let $\Omega$ be an open bounded subset in $\mathbb{R}^n$, lying on one side of its boundary $\partial\Omega$, which is a submanifold of class $C^{2+\beta}$ of $\mathbb{R}^n$. Let $N$ and $\nu$ belong to $\mathbb{R}^+$. We set*

$$A(N,\nu) := \quad \{((a_\alpha)_{|\alpha|\leq 2}, (b_\gamma)_{|\gamma|\leq 1}) : \sum_{|\alpha|\leq 2} \|a_\alpha\|_{C^\beta(\Omega;\mathbb{R})} + \sum_{|\gamma|\leq 1} \|b_\gamma\|_{C^{1+\beta}(\partial\Omega;\mathbb{R})} \leq N,$$

$$\sum_{|\alpha|=2} a_\alpha(x)\xi^\alpha \geq \nu|\xi|^2 \quad \forall(x,\xi) \in \overline{\Omega} \times \mathbb{R}^n, \sum_{\gamma=1} b_\gamma(x')\nu_\gamma(x') \geq \nu \quad \forall(x',\xi) \in \partial\Omega\}.$$

*Then the constants $C(T_0, A, B)$ and $C(T_0, A, B, \theta)$ appearing in the statement of Lemma 2.2 can be taken independently of $A$ and $B$ if $((a_\alpha)_{|\alpha|\leq 2}, (b_\gamma)_{|\gamma|\leq 1}) \in A(N,\nu)$.*

*Proof* Let $((a^0_\alpha)_{|\alpha|\leq 2}, (b^0_\gamma)_{|\gamma|\leq 1}) \in A(N,\nu)$. Then, by Remark 2.4, there exists $d$ in $\mathbb{R}^+$, such that the constants $C(T_0, A, B)$ and $C(T_0, A, B, \theta)$ can be chosen independently of $((a_\alpha)_{|\alpha|\leq 2}, (b_\gamma)_{|\gamma|\leq 1}) \in A(N,\nu)$, in case

$$\sum_{|\alpha|=2} \|a_\alpha - a^0_\alpha\|_{C(\Omega)} + \sum_{|\gamma|=1} \|b_\gamma - b^0_\gamma\|_{C(\Omega)} < d.$$

Now we observe that, by the theorem of Ascoli-Arzelà, $A(N,\nu)$ is compact in $C(\Omega)^{n^2+n+1} \times C(\partial\Omega)^{n+1}$ (we recall, that, for example, a bounded sequence in $C^{\beta/2,1+\beta}((0,T) \times \partial\Omega)$ contains a subsequence uniformly converging to an element of $C^{\beta/2,1+\beta}((0,T) \times \partial\Omega)$). So it can be covered by a finite number of balls of the described type. The conclusion follows.

□

**Theorem 2.6.** *Consider the system (2.1), with the conditions (AF1)-(AF3). Then:*

*(I) if $f \in C^{\beta/2,\beta}((0,T) \times \Omega)$, $h \in C^{\beta/2,1+\beta}((0,T) \times \partial\Omega)$, $u_0 \in C^{2+\beta}(\Omega)$ and*

$$A(0,x',D_x)u_0(x') + f(0,x') = -B(0,x',D_x)u_0(x') + h(0,x') \quad \forall x' \in \partial\Omega,$$

*(2.1) has a unique solution $u$ in $C^{1+\beta/2,2+\beta}((0,T) \times \Omega)$, with $D_t u_{|(0,T)\times\partial\Omega} \in B((0,T); C^{1+\beta}(\partial\Omega))$.*

*Let $T_0 \in \mathbb{R}^+$ and assume that $0 < T \leq T_0$. Then:*

*(II) there exists $C(T_0, N, \nu)$ in $\mathbb{R}^+$, such that*

$$\|u\|_{C^{1+\beta/2,2+\beta}((0,T)\times\Omega))} + \|D_t u_{|(0,T)\times\partial\Omega}\|_{B((0,T);C^{1+\beta}(\partial\Omega))} + \|u\|_{C^{\beta/2}((0,T);C^2(\Omega))} + \|u\|_{C^{\frac{1+\beta}{2}}((0,T);C^1(\Omega))}$$

$$\leq C(T_0, N, \nu)(\|f\|_{C^{\beta/2,\beta}((0,T)\times\Omega)} + \|u_0\|_{C^{2+\beta}(\Omega)} + \|h\|_{C^{\beta/2,1+\beta}((0,T)\times\partial\Omega)}).$$

*(III) Suppose that $u_0 = 0$. Then, if $0 \leq \theta \leq 1$,*

$$\|u\|_{C^\theta((0,T);C(\Omega))} \leq C(T_0, N, \nu)T^{1-\theta}(\|f\|_{C^{\beta/2,\beta}((0,T)\times\Omega)} + \|h\|_{C^{\beta/2,1+\beta}((0,T)\times\partial\Omega)}), \tag{2.23}$$

*if $0 \leq \theta \leq 2 + \beta$,*

$$\|u\|_{B((0,T);C^\theta(\Omega))} \leq C(T_0, N, \nu, \theta)T^{\frac{2+\beta-\theta}{2+\beta}}(\|f\|_{C^{\beta/2,\beta}((0,T)\times\Omega)} + \|h\|_{C^{\beta/2,1+\beta}((0,T)\times\partial\Omega)}). \tag{2.24}$$

*(IV) Again supposing $u_0 = 0$, we have*

$$\|u\|_{C^{\beta/2}((0,T);C^1(\Omega))} \leq C(T_0, N, \nu)T^{1/2}(\|f\|_{C^{\beta/2,\beta}((0,T)\times\Omega)} + \|h\|_{C^{\beta/2,1+\beta}((0,T)\times\partial\Omega)}).$$

*Proof* Let $t_0 \in [0,T)$. We consider the system

$$\begin{cases} D_t v(t,x) - A(t_0+t,x,D_x)v(t,x) \\ = D_t v(t,x) - A(t_0,x,D_x)v(t,x) - [A(t_0+t,x,D_x) - A(t_0,x,D_x)]v(t,x) + \phi(t,x), \\ t \in (0,\tau), x \in \Omega, \\ D_t v(t,x') + B(t_0+t,x',D_x)v(t,x') \\ = D_t v(t,x') + B(t_0,x',D_x)v(t,x') + [B(t_0+t,x',D_x) - B(t_0,x',D_x)]v(t,x') = \psi(t,x'), \\ t \in (0,\tau), x' \in \partial\Omega, \\ v(0,x) = v_0(x), \quad x \in \Omega, \end{cases} \tag{2.25}$$

We have, for $|\alpha| = 2$, $t \in [0, T - t_0]$, $x \in \Omega$, $|\gamma| = 1$, $x' \in \partial\Omega$,

$$|a_\alpha(t_0 + t, x) - a_\alpha(t_0, x)| \le N t^{\beta/2}, |b_\gamma(t_0 + t, x') - b_\gamma(t_0, x')| \le N t^{\beta/2},$$

So, by Lemma 2.3, Remark 2.4 and Lemma 2.3, there exists $\tau_0 \in \mathbb{R}^+$, independent of $t_0$ in $[0, T)$, such that, if $0 < \tau \le \tau_0 \wedge (T - \tau_0)$, $\phi \in C^{\beta/2,\beta}((0,\tau) \times \Omega))$, $\psi \in C^{\beta/2,1+\beta}((0,\tau) \times \partial\Omega)$, $v_0 \in C^{2+\beta}(\Omega)$,

$$A(t_0, x', D_x) v_0(x') + \phi(0, x') = -B(t_0, x', D_x) v_0(x') + \psi(0, x') \quad \forall x' \in \partial\Omega,$$

(2.25) has a unique solution $v$ in $C^{1+\beta/2,2+\beta}((0,\tau) \times \Omega)$, with $D_t v_{|(0,\tau)\times\partial\Omega} \in B((0,\tau); C^{1+\beta}(\partial\Omega))$. Moreover, there exists $C(\tau_0, N, \nu)$ in $\mathbb{R}^+$, such that

$$\|v\|_{C^{1+\beta/2,2+\beta}((0,\tau)\times\Omega))} + \|D_t v_{|(0,\tau)\times\partial\Omega}\|_{B((0,\tau);C^{1+\beta}(\partial\Omega))} + \|v\|_{C^{\beta/2}((0,\tau);C^2(\Omega))} + \|v\|_{C^{\frac{1+\beta}{2}}((0,\tau);C^1(\Omega))}$$

$$\le C(\tau_0, N, \nu)(\|\phi\|_{C^{\beta/2,\beta}((0,\tau)\times\Omega)} + \|v_0\|_{C^{2+\beta}(\Omega)} + \|\psi\|_{C^{\beta/2,1+\beta}((0,\tau)\times\partial\Omega)}).$$

So we can begin by constructing a solution $v$ in $(0, \tau_0 \wedge T) \times \Omega$. In case $\tau_0 < T$, we can consider (2.25) with $t_0 = \tau_0$, $\phi(t, x) = f(\tau + t, x)$, $\psi(t, x') = h(\tau + t, x')$, $v_0 = u(\tau_0, \cdot)$. In a finite number of steps we get (I) and (II). (III) and (IV) can be obtained as in the proof of Lemma 2.2.

□

# 3 Quasilinear problems

We are going to discuss system (1.1).

We have:

**Lemma 3.1.** *Let $\Omega$ be an open bounded subset of $\mathbb{R}^n$ and let $a : [0, T] \times \overline{\Omega} \times \mathbb{R}^{n+1} \to \mathbb{R}$. We assume the following:*

*(a) $\forall (t, x) \in [0, T] \times \overline{\Omega}$ $a(t, x, \cdot, \cdot) \in \mathcal{C}^1(\mathbb{R}^{n+1})$;*

*(b) for some $\beta \in (0, 1)$, $\forall R \in \mathbb{R}^+$ there exists $C(R) \in \mathbb{R}^+$ such that, $\forall s, t \in [0, T]$, $\forall x, y \in \overline{\Omega}$, $\forall u, v \in \mathbb{R}$, $\forall p, q \in \mathbb{R}^n$,*

$$|a(t, x, u, p) - a(s, y, v, q)| + |\nabla_{u,p} a(t, x, u, p) - \nabla_{u,p} a(s, y, v, q)|$$

$$\le C(R)(|t - s|^{\beta/2} + |x - y|^\beta + |u - v| + |p - q|)$$

*whenever $|u| + |v| + |p| + |q| \le R$. For $\tau \in (0, T]$ and $u : [0, \tau] \times \overline{\Omega} \to \mathbb{R}$, define*

$$A(u)(t, x) := a(t, x, u(t, x), \nabla_x u(t, x))$$

*whenever the second term has a meaning. Then:*

*(I) if $u \in C^{\beta/2}((0,\tau), C^1(\Omega)) \cap B((0,\tau); C^{1+\beta}(\Omega))$, $R \in \mathbb{R}^+$ and*

$$\max\{\|u\|_{C^{\beta/2}((0,\tau),C^1(\Omega))}, \|u\|_{B((0,\tau),C^{1+\beta}(\Omega))}\} \le R,$$

*$A(u) \in C^{\beta/2,\beta}((0,\tau) \times \Omega)$ and $\|A(u)\|_{C^{\beta/2,\beta}((0,\tau)\times\Omega)} \le C_1(R)$;*

*(II) if $u, v \in C^{\beta/2}((0,\tau), C^1(\Omega)) \cap B((0,\tau); C^{1+\beta}(\Omega))$ and*

$$\max\{\|u\|_{C^{\beta/2}((0,\tau),C^1(\Omega))}, \|u\|_{B((0,\tau),C^{1+\beta}(\Omega))}, \|v\|_{C^{\beta/2}((0,\tau),C^1(\Omega))}, \|v\|_{B((0,\tau),C^{1+\beta}(\Omega))}\} \le R,$$

*then*

$$\|A(u) - A(v)\|_{C^{\beta/2,\beta}((0,\tau)\times\Omega)} \le C_2(R)(\|u - v\|_{C^{\beta/2}((0,\tau),C^1(\Omega))} + \|u - v\|_{B((0,\tau),C^{1+\beta}(\Omega))}).$$

*Proof.* (I) If $(t, x) \in (0, T) \times \Omega$,

$$|A(u)(t, x)| \le |a(t, x, u(t, x), \nabla_x u(t, x)) - a(t, x, 0, 0)| + |a(t, x, 0, 0)|$$

$$\le \|a(\cdot, \cdot, 0, 0)\|_{C((0,T)\times\Omega)} + C(R)(|u(t, x)| + |\nabla_x u(t, x)|) \le C_3(R).$$

Moreover, if $s,t\in(0,\tau)$ and $x\in\Omega$,

$$|a(t,x,u(t,x),\nabla_x u(t,x))-a(s,x,u(s,x),\nabla_x u(s,x))|$$

$$\leq C(R)(|t-s|^{\beta/2}+|u(t,x)-u(s,x)|+|\nabla_x u(t,x)-\nabla_x u(s,x)|)$$

$$\leq C(R)(|t-s|^{\beta/2}+(t-s)^{\beta/2}\|u\|_{C^{\beta/2}((0,T);C(\Omega))}+(t-s)^{\beta/2}\|u\|_{C^{\beta/2}((0,T);C^1(\Omega))})$$

$$\leq C_4(R)|t-s|^{\beta/2}.$$

Analogously, one can show that $\|A(u)\|_{B((0,\tau);C^\beta(\Omega))}\leq C_5(R)$.

(II) If $(t,x)\in(0,\tau)\times\Omega$,

$$|A(u)(t,x)-A(v)(t,x)|\leq C(R)(|u(t,x)-v(t,x)|+|\nabla_x u(t,x)-\nabla_x v(t,x)|)$$

$$\leq 2C(R)\|u-v\|_{C((0,\tau);C^1(\Omega))}\leq 2C(R)\|u-v\|_{C^{\beta/2}((0,\tau);C^1(\Omega))}.$$

If $t,s\in(0,\tau)$ and $x\in\Omega$,

$$|(A(u)(t,x)-A(v)(t,x))-(A(u)(s,x)-A(v)(s,x))|$$

$$=|\textstyle\int_0^1\nabla_{u,p}a(t,x,v(t,x)+r(u(t,x)-v(t,x)),\nabla_x v(t,x)+r(\nabla_x u(t,x)-\nabla_x v(t,x)))dr$$

$$\cdot(u(t,x)-v(t,x),\nabla_x u(t,x)-\nabla_x v(t,x))$$

$$-\textstyle\int_0^1\nabla_{u,p}a(s,x,v(s,x)+r(u(s,x)-v(s,x)),\nabla_x v(s,x)+r(\nabla_x u(s,x)-\nabla_x v(s,x)))dr$$

$$\cdot(u(s,x)-v(s,x),\nabla_x u(s,x)-\nabla_x v(s,x))|$$

$$\leq|\textstyle\int_0^1\nabla_{u,p}a(t,x,v(t,x)+r(u(t,x)-v(t,x)),\nabla_x v(t,x)+r(\nabla_x u(t,x)-\nabla_x v(t,x))dr|$$

$$\times|(u(t,x)-v(t,x)-(u(s,x)-v(s,x)),\nabla_x u(t,x)-\nabla_x v(t,x)-(\nabla_x u(s,x)-\nabla_x v(s,x)))|$$

$$+\textstyle\int_0^1|\nabla_{u,p}a(t,x,v(t,x)+r(u(t,x)-v(t,x)),\nabla_x v(t,x)+r(\nabla_x u(t,x)-\nabla_x v(t,x))$$

$$-\nabla_{u,p}a(s,x,v(s,x)+r(u(s,x)-v(s,x)),\nabla_x v(s,x)+r(\nabla_x u(s,x)-\nabla_x v(s,x))|dr$$

$$\times|(u(s,x)-v(s,x),\nabla_x u(s,x)-\nabla_x v(s,x))|=I+J$$

and

$$I\leq C_5(R)(|u(t,x)-v(t,x)-(u(s,x)-v(s,x))|$$

$$+|\nabla_x u(t,x)-\nabla_x v(t,x)-(\nabla_x u(s,x)-\nabla_x v(s,x))|)$$

$$\leq 2C_5(R)|t-s|^{\beta/2}\|u-v\|_{C^{\beta/2}((0,\tau);C^1(\Omega))},$$

$$J\leq C_6(R)(|t-s|^{\beta/2}+|u(t,x)-u(s,x)|+|v(t,x)-v(s,x)|$$

$$+|\nabla_x u(t,x)-\nabla_x u(s,x)|+|\nabla_x v(t,x)-\nabla_x v(s,x)|)$$

$$\times(|u(s,x)-v(s,x)|+|\nabla_x u(s,x)-\nabla_x v(s,x)|)$$

$$\leq C_7(R)|t-s|^{\beta/2}(1+\|u\|_{C^{\beta/2}((0,\tau);C^1(\Omega))}+\|v\|_{C^{\beta/2}((0,\tau);C^1(\Omega))})\|u-v\|_{C((0,\tau);C^1(\Omega))}.$$

So

$$[A(u)-A(v)]_{C^{\beta/2}((0,\tau);C(\Omega))}\leq C_8(R)\|u-v\|_{C^{\beta/2}((0,\tau);C^1(\Omega))}.$$

Analogously, one can show that

$$\|A(u)-A(v)\|_{B((0,\tau);C^\beta(\Omega))}\leq C_9(R)\|u-v\|_{B((0,\tau);C^{1+\beta}(\Omega))}.$$

□

**Lemma 3.2.** *Let $\Omega$ be an open bounded subset of $\mathbb{R}^n$, $\beta \in (0,1))$ and let $b : [0,T] \times \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$. We assume the following:*

*(a) $b$ is continuous together with its partial derivatives $D_{x_j} b$, $D_u b$, $D^2_{x_j u} b$, $D^2_u b$ $(1 \le j \le n)$;*

*(b) $\forall R \in \mathbb{R}^+$ there exists $C(R) \in \mathbb{R}^+$ such that, $\forall t, s \in [0,T]$, $\forall x \in \overline{\Omega}$, $\forall u, v \in \mathbb{R}$ with $|u| + |v| \le R$,*

$$|b(t,x,u) - b(s,x,v)| + |D_u b(t,x,u) - D_u b(s,x,v)| \le C(R)(|t-s|^{\beta/2} + |u-v|).$$

*(c) $\forall R \in \mathbb{R}^+$ there exists $C(R) \in \mathbb{R}^+$ such that, $\forall t \in [0,T]$, $\forall x, y \in \overline{\Omega}$, $\forall u, v \in \mathbb{R}$ with $|u| + |v| \le R$,*

$$|\nabla_{x,u} b(t,x,u) - \nabla_{x,u} b(t,y,v)| + |D_u \nabla_{x,u} b(t,x,u) - D_u \nabla_{x,u} b(t,y,v)| \le C(R)(|x-y|^{\beta} + |u-v|).$$

*For $\tau \in (0,T]$ and $u : [0,\tau] \times \overline{\Omega} \to \mathbb{R}$, define*

$$B(u)(t,x) := b(t,x,u(t,x)).$$

*Then:*

*(I) if $u \in C^{\beta/2, 1+\beta}((0,\tau) \times \Omega))$, $R \in \mathbb{R}^+$ and*

$$\|u\|_{C^{\beta/2,1+\beta}((0,\tau)\times\Omega)} \le R,$$

*$B(u) \in C^{\beta/2,1+\beta}((0,\tau) \times \Omega)$ and $\|B(u)\|_{C^{\beta/2,1+\beta}((0,\tau)\times\Omega)} \le C_1(R)$;*

*(II) if $u, v \in C^{\beta/2,1+\beta}((0,\tau) \times \Omega)$ and*

$$\max\{\|u\|_{C^{\beta/2,1+\beta}((0,\tau)\times\Omega)}, \|v\|_{C^{\beta/2,1+\beta}((0,\tau)\times\Omega)}\} \le R,$$

*then*

$$\|B(u) - B(v)\|_{C^{\beta/2,1+\beta}((0,\tau)\times\Omega)} \le C_2(R)\|u - v\|_{C^{\beta/2,1+\beta}((0,\tau)\times\Omega))}.$$

*Proof.* It follows the lines of the proof of Lemma 3.1; of course, one should use the elementary formula

$$D_{x_j} B(u)(t,x) = D_{x_j} b(t,x,u(t,x)) + D_u b(t,x,u(t,x)) D_{x_j} u(t,x) \quad (1 \le j \le n).$$

□

Now we are able to study system (1.1). We introduce the following assumptions:

*(AG1) $\beta \in (0,1)$, $\Omega$ is an open bounded subset of $\mathbb{R}^n$, with boundary $\partial\Omega$ of class $C^{2+\beta}$;*

*(AG2) $T \in \mathbb{R}^+$ and $\forall \alpha \in \mathbb{N}_0^n$, with $|\alpha| = 2$, $a_\alpha, f : [0,T] \times \overline{\Omega} \times \mathbb{R}^{n+1} \to \mathbb{R}$; $\forall (t,x) \in [0,T] \times \overline{\Omega}$ $a_\alpha(t,x,\cdot,\cdot), f(t,x,\cdot,\cdot) \in \mathcal{C}^1(\mathbb{R}^{n+1})$; $\forall R \in \mathbb{R}^+$ there exists $C(R) \in \mathbb{R}^+$ such that, $\forall s, t \in [0,T]$, $\forall x, y \in \overline{\Omega}$, $\forall u, v \in \mathbb{R}$, $\forall p, q \in \mathbb{R}^n$,*

$$\sum_{|\alpha|=2}(|a_\alpha(t,x,u,p) - a_\alpha(s,y,v,q)| + |\nabla_{u,p} a_\alpha(t,x,u,p) - \nabla_{u,p} a_\alpha(s,y,v,q)|)$$

$$+|f(t,x,u,p) - f(s,y,v,q)| + |\nabla_{u,p} f(t,x,u,p) - \nabla_{u,p} f(s,y,v,q)|$$

$$\le C(R)(|t-s|^{\beta/2} + |x-y|^\beta + |u-v| + |p-q|).$$

*(AG3) $\forall j \in \{1,\dots,n\}$ $b_j, h : [0,T] \times \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$; they are continuous, together with their derivatives $D_{x_i} b_j$, $D_u b_j$, $D^2_{x_i u} b_j$, $D^2_u b_j$, $D_{x_i} h$, $D_u h$, $D^2_{x_i u} h$, $D^2_u h$ $(1 \le i \le n)$; $\forall R \in \mathbb{R}^+$ there exists $C(R) \in \mathbb{R}^+$ such that, $\forall t, s \in [0,T]$, $\forall x \in \overline{\Omega}$, $\forall u, v \in \mathbb{R}$ with $|u| + |v| \le R$,*

$$\sum_{j=1}^n (|b_j(t,x,u) - b_j(s,x,v)|$$

$$+|D_u b_j(t,x,u) - D_u b_j(s,x,v)|) + |h(t,x,u) - h(s,x,v)| + |D_u h(t,x,u) - D_u h(s,x,v)|$$

$$\le C(R)(|t-s|^{\beta/2} + |u-v|);$$

*$\forall R \in \mathbb{R}^+$ there exists $C(R) \in \mathbb{R}^+$ such that, $\forall t \in [0,T]$, $\forall x, y \in \overline{\Omega}$, $\forall u, v \in \mathbb{R}$ with $|u| + |v| \le R$,*

$$\sum_{j=1}^n (|\nabla_{x,u} b_j(t,x,u) - \nabla_{x,u} b_j(t,y,v)| + |D_u \nabla_{x,u} b_j(t,x,u) - D_u \nabla_{x,u} b_j(t,y,v)|)$$

$$+|\nabla_{x,u} h(t,x,u) - \nabla_{x,u} h(t,y,v)| + |D_u \nabla_{x,u} h(t,x,u) - D_u \nabla_{x,u} h(t,y,v)| \le C(R)(|x-y|^\beta + |u-v|).$$

*(AG4)* $\forall (t,x,u,p) \in [0,T] \times \overline{\Omega} \times \mathbb{R}^{n+1}$ *there exists* $\nu(t,x,u,p) \in \mathbb{R}^+$ *such that*

$$\sum_{|\alpha|=2} a_\alpha(t,x,u,p)\xi^\alpha \geq \nu(t,x,u,p)|\xi|^2 \quad \forall \xi \in \mathbb{R}^n;$$

$\forall (t,x',u) \in [0,T] \times \partial\Omega \times \mathbb{R}$ *there exists* $\nu(t,x',u) \in \mathbb{R}^+$ *such that*

$$\sum_{j=1}^n b_j(t,x',u)\nu_j(x') \geq \nu(t,x',u).$$

**Lemma 3.3.** *Assume that (AG1)-(AG4) hold. Let* $u_0 \in C^{2+\beta}(\Omega)$ *be such that*

$$\sum_{|\alpha|=2} a_\alpha(0,x',u_0(x'),\nabla_x u_0(x'))D_x^\alpha u_0(x') + f(0,x',u_0(x'),\nabla_x u_0(x'))$$
$$= -\sum_{j=1}^n b_j(0,x',u_0(x'))D_{x_j} u_0(x') + h(0,x',u_0(x')), \quad \forall x' \in \partial\Omega.$$

*Consider the system*

$$\begin{cases} D_t U_0(t,x) = \sum_{|\alpha|=2} a_\alpha(t,x,u_0(x),\nabla_x u_0(x))D_x^\alpha U_0(t,x) + f(t,x,u_0(x),\nabla_x u_0(x)), & t \in (0,T), x \in \Omega, \\ D_t U_0(t,x') + \sum_{j=1}^n b_j(t,x',u_0(x'))D_{x_j} U_0(t,x') = h(t,x',u_0(x')), & t \in (0,T), x' \in \partial\Omega, \\ U_0(0,x) = u_0(x), & x \in \Omega. \end{cases} \tag{3.1}$$

*Then (3.1) has a unique solution* $U_0$ *belonging to* $C^{1+\beta/2,2+\beta}((0,T) \times \Omega)$*, with* $D_t U_{0|(0,T)\times\partial\Omega} \in C^{\beta/2,1+\beta}((0,T) \times \partial\Omega)$*.*

**Proposition 3.4.** *Consider the system (1.1), with the assumptions* $(AG1) - (AG3)$*. Let* $u_0 \in C^{2+\beta}(\Omega)$ *be such that*

$$\sum_{|\alpha|=2} a_\alpha(0,x',u_0(x'),\nabla_x u_0(x'))D_x^\alpha u_0(x') + f(0,x',u_0(x'))$$
$$= -\sum_{j=1}^n b_j(0,x',u_0(x'))D_{x_j} u_0(x') + h(0,x',u_0(x')), \quad \forall x' \in \partial\Omega.$$

*Let* $R \in \mathbb{R}^+$*. Then there exists* $\tau(R) \in (0,T]$ *such that, if* $0 < \tau \leq \tau(R)$*, (1.1) has a unique solution* $u$ *in* $C^{1+\beta/2,2+\beta}((0,\tau) \times \Omega)$*, with* $D_t u_{|(0,\tau)\times\partial\Omega} \in B((0,\tau); C^{1+\beta}(\partial\Omega))$*, satisfying*

$$\|u - U_0\|_{C^{\beta/2}((0,\tau);C^1(\Omega))} + \|u - U_0\|_{B((0,\tau);C^{1+\beta}(\Omega))} \leq R.$$

*Proof.* We consider, for $\tau \in (0,T]$, $R \in \mathbb{R}$, the set

$$\begin{aligned} \mathcal{M}(\tau,R) := \quad & \{U \in C^{\beta/2}((0,\tau);C^1(\Omega)) \cap B((0,\tau);C^{1+\beta}(\Omega)) : U(0,\cdot) = u_0, \\ & \|u - U_0\|_{C^{\beta/2}((0,\tau);C^1(\Omega))} + \|u - U_0\|_{B((0,\tau);C^{1+\beta}(\Omega))} \leq R\} \end{aligned}$$

which is a complete metric space with the distance

$$d(U,V) := \|U - V\|_{C^{\beta/2}((0,\tau);C^1(\Omega))} + \|U - V\|_{B((0,\tau);C^{1+\beta}(\Omega))}.$$

If $U \in \mathcal{M}(\tau,R)$, we consider the system

$$\begin{cases} D_t u(t,x) = \sum_{|\alpha|=2} a_\alpha(t,x,U(t,x),\nabla_x U(t,x))D_x^\alpha u(t,x) + f(t,x,U(t,x),\nabla_x U(t,x)), & t \in (0,\tau), x \in \Omega, \\ D_t u(t,x') + \sum_{j=1}^n b_j(t,x',U(t,x'))D_{x_j} u(t,x') = h(t,x,U(t,x)), & t \in (0,\tau), x' \in \partial\Omega, \\ u(0,x) = u_0(x), & x \in \Omega. \end{cases} \tag{3.2}$$

Then, applying Lemmata 3.1 and 3.2 and Theorem 2.6, we obtain that (3.2) has a unique solution $u = u(U)$ in $C^{1+\beta/2,2+\beta}((0,\tau) \times \Omega)$, with $D_t u_{|(0,\tau)\times\partial\Omega} \in B((0,\tau); C^{1+\beta}(\partial\Omega))$. Moreover, there exists $C_1(R)$ in $\mathbb{R}^+$, independent of $\tau$ and $U$, such that

$$\begin{aligned} &\|u\|_{C^{1+\beta/2,2+\beta}((0,T)\times\Omega))} + \|D_t u_{|(0,T)\times\partial\Omega}\|_{B((0,T);C^{1+\beta}(\partial\Omega))} + \|u\|_{C^{\beta/2}((0,T);C^2(\Omega))} \\ &\quad + \|u\|_{C^{\frac{1+\beta}{2}}((0,T);C^1(\Omega))} \leq C_1(R). \end{aligned} \tag{3.3}$$

Hence, we have

$$\begin{cases} D_t(u-U_0)(t,x) = \sum_{|\alpha|=2} a_\alpha(t,x,u_0(x),\nabla_x u_0(x)) D_x^\alpha (u-U_0)(t,x) \\ +\sum_{|\alpha|=2}[a_\alpha(t,x,U(t,x),\nabla_x U(t,x)) - a_\alpha(t,x,u_0(x),\nabla_x u_0(x))] D_x^\alpha u(t,x) \\ +f(t,x,U(t,x)) - f(t,x,u_0(x)), & t\in(0,\tau), x\in\Omega, \\ D_t(u-U_0)(t,x') + \sum_{j=1}^n b_j(t,x',u_0(x')) D_{x_j}(u-U_0)(t,x') \\ +\sum_{j=1}^n [b_j(t,x',U(t,x')) - b_j(t,x',u_0(x'))] D_{x_j} u(t,x') \\ = h(t,x',U(t,x')) - h(t,x,u_0(x')), & t\in(0,\tau), x'\in\partial\Omega, \\ (u-U_0)(0,x) = 0, & x\in\Omega. \end{cases}$$

So, from Theorem 2.6 (III)-(IV), (3.3) and Lemmata 1.1-3.1, we obtain

$$d(u,U_0) \le C_2(R)\tau^{1/2}.$$

Choosing $\tau$ such that $C_2(R)\tau^{1/2} \le R$, we obtain that $U \to u(U)$ maps $\mathcal{M}(\tau,R)$ into itself. Next, we show that, if $\tau$ is sufficiently small, it is a contraction in $\mathcal{M}(\tau,R)$. Let $U,V \in \mathcal{M}(\tau,R)$. We indicate with $u$ and $v$ the corresponding solutions to (3.2). Then we have

$$\begin{cases} D_t(u-v)(t,x) = \sum_{|\alpha|=2} a_\alpha(t,x,V(t,x),\nabla_x V(t,x)) D_x^\alpha (u-v)(t,x) \\ +\sum_{|\alpha|=2}[a_\alpha(t,x,U(t,x),\nabla_x U(t,x)) - a_\alpha(t,x,V(t,x),\nabla_x V(t,x))] D_x^\alpha u(t,x) \\ +f(t,x,U(t,x)) - f(t,x,V(t,x)), & t\in(0,\tau), x\in\Omega, \\ D_t(u-v)(t,x') + \sum_{j=1}^n b_j(t,x',V(t,x')) D_{x_j}(u-v)(t,x') \\ +\sum_{j=1}^n [b_j(t,x',U(t,x')) - b_j(t,x',V(t,x'))] D_{x_j} u(t,x') \\ = h(t,x',U(t,x')) - h(t,x,V(t,x')), & t\in(0,\tau), x'\in\partial\Omega, \\ (u-v)(0,x) = 0, & x\in\Omega. \end{cases}$$

Arguing as before, we obtain

$$\begin{gathered} d(u,v) \\ \le C_3(R)\tau^{1/2}\Big(\sum_{|\alpha|=2} \|a_\alpha(t,x,U(t,x),\nabla_x U(t,x)) - a_\alpha(t,x,V(t,x),\nabla_x V(t,x))] D_x^\alpha u\|_{C^{\beta/2,\beta}((0,\tau)\times\Omega)} \\ +\|f(t,x,U(t,x)) - f(t,x,V(t,x))\|_{C^{\beta/2,\beta}((0,\tau)\times\Omega)} + \\ +\sum_{j=1}^n \|[b_j(t,x',U(t,x')) - b_j(t,x',V(t,x'))] D_{x_j} u(t,x')\|_{C^{\beta/2,1+\beta}((0,\tau)\times\partial\Omega)} + \\ +\|h(t,x',U(t,x')) - h(t,x,V(t,x'))\|_{C^{\beta/2,1+\beta}((0,\tau)\times\partial\Omega)}\Big). \end{gathered}$$

We have, for $|\alpha| = 2$

$$\begin{gathered} \sum_{|\alpha|=2} \|a_\alpha(t,x,U(t,x),\nabla_x U(t,x)) - a_\alpha(t,x,V(t,x),\nabla_x V(t,x))] D_x^\alpha u\|_{C^{\beta/2,\beta}((0,\tau)\times\Omega)} \\ \le C_1 \sum_{|\alpha|=2} \|a_\alpha(t,x,U(t,x),\nabla_x U(t,x)) - a_\alpha(t,x,V(t,x),\nabla_x V(t,x))\|_{C^{\beta/2,\beta}((0,\tau)\times\Omega)} \\ \times(\|u\|_{C^{\beta/2}((0,\tau);C^2)} + \|u\|_{C^{\beta/2,2+\beta}((0,\tau)\times\Omega)}) \\ \le C_4(R)d(U,V), \\ \|f(t,x,U(t,x)) - f(t,x,V(t,x))\|_{C^{\beta/2,\beta}((0,\tau)\times\Omega)} \le C_5(R)d(U,V), \end{gathered}$$

$$\sum_{j=1}^n \|[b_j(t,x',U(t,x')) - b_j(t,x',V(t,x'))]D_{x_j}u(t,x')\|_{C^{\beta/2,1+\beta}((0,\tau)\times\partial\Omega)}$$

$$\leq C_2 \sum_{j=1}^n \|[b_j(t,x',U(t,x')) - b_j(t,x',V(t,x'))]\|_{C^{\beta/2,1+\beta}((0,\tau)\times\partial\Omega)} \|D_{x_j}u\|_{C^{\beta/2,1+\beta}((0,\tau)\times\partial\Omega)}$$

$$\leq C_3 \sum_{j=1}^n \|[b_j(t,x',U(t,x')) - b_j(t,x',V(t,x'))]\|_{C^{\beta/2,1+\beta}((0,\tau)\times\partial\Omega)}$$

$$\times (\|u\|_{C^{\frac{1+\beta}{2}}((0,\tau);C^1(\Omega))} + \|u\|_{C^{\beta/2,2+\beta}((0,\tau)\times\Omega)})$$

$$\leq C_5(R)\|U-V\|_{C^{\beta/2,1+\beta}((0,\tau)\times\partial\Omega)} \leq C_5(R)d(U,V).$$

We conclude that

$$d(u,v) \leq C_6(R)\tau^{1/2}d(U,V),$$

implying that $U \to u(U)$ is a contraction if $C_6(R)\tau^{1/2} < 1$.

So the conclusion follows from the contraction mapping theorem.

□

We deduce the following

**Theorem 3.5.** *Consider system (1.1), with the conditions (AG1)-(AG4). Let $u_0 \in C^{2+\beta}(\Omega)$ be such that*

$$\begin{aligned} &\sum_{|\alpha|=2} a_\alpha(0,x',u_0(x'),\nabla_x u_0(x'))D_x^\alpha u_0(x') + f(0,x',u_0(x')) \\ &= -\sum_{j=1}^n b_j(0,x',u_0(x'))D_{x_j}u_0(x') + h(0,x',u_0(x')), \quad \forall x' \in \partial\Omega. \end{aligned} \tag{3.4}$$

*Then there exists $\tau \in (0,T]$ such that (1.1) admits a unique solution $u$ in $C^{1+\beta/2,2+\beta}((0,\tau)\times\Omega)$, with $D_t u_{|(0,\tau)\times\partial\Omega} \in B((0,\tau);C^{1+\beta}(\partial\Omega))$.*

*Proof.* The existence follows from Proposition 3.4.

Concerning the uniqueness, let $u$ and $v$ be different solutions in $(0,\tau)\times\Omega$. Let $\tau_1 := \inf\{t \in [0,\tau] : u(t,\cdot) \neq v(t,\cdot)\}$. Then $0 \leq \tau_1 < \tau$ and $u(t,\cdot) = v(t,\cdot)$ $\forall t \in [0,\tau_1]$. We set $u_1 := u(\tau_1,\cdot) = v(\tau_1,\cdot)$. We consider the system

$$\begin{cases} D_t z(t,x) = \sum_{|\alpha|=2} a_\alpha(\tau_1+t,x,z(t,x),\nabla_x z(t,x))D_x^\alpha z(t,x) + f(\tau_1+t,x,z(t,x),\nabla_x z(t,x)), & t \geq 0, x \in \Omega, \\ D_t z(t,x') + \sum_{j=1}^n b_j(\tau_1+t,x',z(t,x'))D_{x_j}z(t,x') = h(\tau_1+t,x',z(t,x')), & t \geq 0, x' \in \partial\Omega, \\ z(0,x) = u_1(x), & x \in \Omega. \end{cases} \tag{3.5}$$

Then $u_1 = u(\tau_1,\cdot) \in C^{2+\beta}(\Omega)$. Moreover, if $x' \in \partial\Omega$,

$$\sum_{|\alpha|=2} a_\alpha(\tau_1,x',u_1(x'),\nabla_x u_1(x'))D_x^\alpha u_1(x') + f(\tau_1,x',u_1(x'),\nabla_x u_1(x'))$$

$$= \sum_{|\alpha|=2} a_\alpha(\tau_1,x',u(\tau_1,x'),\nabla_x u(\tau_1,x'))D_x^\alpha u(\tau_1,x') + f(\tau_1,x',u(\tau_1,x'),\nabla_x u(\tau_1,x'))$$

$$= -\sum_{j=1}^n b_j(\tau_1,x',u(\tau_1,x'))D_{x_j}u(\tau_1,x') + h(\tau_1,x',u(\tau_1,x'))$$

$$= -\sum_{j=1}^n b_j(\tau_1,x',u_1(x'))D_{x_j}u_1(x') + h(\tau_1,x',u_1(x')).$$

We indicate with $U_1$ the unique solution in $C^{1+\beta/2,2+\beta}((0,T-\tau_1)\times\Omega)$ with $D_t U_{1|(0,T-\tau_1)\times\partial\Omega} \in B((0,T-\tau_1);C^{1+\beta}(\partial\Omega))$ of

$$\begin{cases} D_t z(t,x) = \sum_{|\alpha|=2} a_\alpha(\tau_1+t,x,u_1(x),\nabla_x u_1(x))D_x^\alpha z(t,x) + f(\tau_1+t,x,u_1(x),\nabla_x u_1(x)), \\ \qquad t \in (0,T-\tau_1), x \in \Omega, \\ D_t z(t,x') + \sum_{j=1}^n b_j(\tau_1+t,x',u_1(x'))D_{x_j}z(t,x') = h(\tau_1+t,x',u_1(x')), \\ \qquad t \in (0,T-\tau_1), x' \in \partial\Omega, \\ z(0,x) = u_1(x), \quad x \in \Omega. \end{cases}$$

existing by Theorem 2.6. Arguing as in the proof of Proposition 3.4, we deduce that, $\forall R \in \mathbb{R}^+$ there exists $\tau(R) \in \mathbb{R}^+$ such that, if $0 < \delta \le \tau(R)$, (3.5) has a unique solution $z$ in $C^{1+\beta/2,2+\beta}((0,\delta)\times\Omega)$, with $D_t z_{|(0,\delta)\times\partial\Omega} \in B((0,\delta); C^{1+\beta}(\partial\Omega))$ and satisfying

$$\|z - U_1\|_{C^{\beta/2}((0,\delta);C^1(\Omega))} + \|z - U_1\|_{B((0,\delta);C^{1+\beta}(\Omega))} \le R.$$

Evidently, $u(\tau_1 + \cdot, \cdot)$ and $v(\tau_1 + \cdot, \cdot)$ both satisfy (3.5) in $(0, \tau - \tau_1) \times \Omega$. Moreover, with the usual method we can see that

$$\lim_{\delta\to 0} \{\|u(\tau_1 + \cdot, \cdot) - U_1\|_{C^{\beta/2}((0,\delta);C^1(\Omega))} + \|u(\tau_1 + \cdot, \cdot) - U_1\|_{B((0,\delta);C^{1+\beta}(\Omega))}$$

$$+\|v(\tau_1 + \cdot, \cdot) - U_1\|_{C^{\beta/2}((0,\delta);C^1(\Omega))} + \|u(\tau_1 + \cdot, \cdot) - U_1\|_{B((0,\delta);C^{1+\beta}(\Omega))}\} = 0$$

We deduce that there exists some $\delta$ in $(0, \tau - \tau_1]$ such that $u$ and $v$ coincide in $[\tau_1, \tau_1 + \delta] \times \Omega$, in contradiction with the definition of $\tau_1$.

□

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